Detailed description of the closing condition and the process for establishing the need for price increases

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1 The closing condition

The second stage of the auction closes when at least one value-maximising feasible combination is inclusive.

Let the vector $\mathbf{s} = (s_{A1}, s_{A2}, s_{A3}, s_B, s_C, s_D, s_E, s_F)$ denote the available supply of lots in categories $A1, A2, A3, B, C, D, E$ and F in the second auction stage (the actual supply of $A1, A2, A3, B$ and C lots will depend on the outcome of the first stage).

Let $\beta_j^k = (P_j^k; \mathbf{q})$ be bid k from bidder $j = 1, ...N$ where N is the number of bidders taking part in the second auction stage, P_j^k is the bid amount and $\mathbf{q}_j^k$ is a vector specifying the bid package (i.e. the number of lots in in each category for which the bidder is prepared to pay the bid amount). Let q_λ denote the number of lots in category λ included in the bid package. K_j denotes the number of bids submitted by bidder j. Notice that the set of bids for a bidder may include a zero bid with $P = 0$ and $q_\lambda = 0 \; \forall \lambda \in \{A1, A2, A3, B, C, D, E, F\}$.

A combination of bids is defined by $\mathbf{x} = (x_1^1, x_1^2, ...x_1^{K_1}, x_2^1, ..., x_N^{K_N})$, where x_j^k is a binary variable that is set equal to 1 if bidder j's k-th bid is included in the combination and zero otherwise.

In order to check whether the closing condition is met, we define the following:

A **feasible combination** is as a combination of bids that satisfies the following conditions:

$$\sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k \mathbf{q}_j^k \leq \mathbf{s} \tag{1.1}$$

$$\sum_{k=1}^{K_j} x_j^k \leq 1 \; \forall j = 1, ..., N \tag{1.2}$$

The **value** of a feasible combination is denoted as:

$$V(\mathbf{x}) = \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k P_j^k + \left(\mathbf{s} - \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k \mathbf{q}_j^k \right) \mathbf{r}$$

where $\mathbf{r}$ is a vector with the reserve prices of the lots in the respective categories.

The set of **value-maximising** feasible combinations, X^*, is:

$$X^* = \left\{ \mathbf{x}^* : \arg\max_{\mathbf{x}} V(\mathbf{x}) \right\}$$

subject to conditions 1.1 and 1.2.

A feasible combination $\mathbf{x}$ is **inclusive** if:

$$\sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k = N$$

Therefore, the second auction stage closes if:

$$\exists\, \mathbf{x} \in X^* : \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k = N$$

2 Identifying lot categories for which prices need to increase

If the second auction stage does not close, then none of the value-maximising feasible combinations is inclusive. Therefore, in each of the value-maximising feasible combination there is at least one bidder for whom no bid is included in the combination. However, this may be because such a bid is tied with leaving the lots requested by the bidder unsold. We are interested only in those value-maximising feasible combinations in which it would not be possible to include bids from one or more of these bidders without discarding any bids from other bidders included in the combination. Say that $\mathbf{x}$ dominates $\hat{\mathbf{x}}$ if $x_j^k \geq \hat{x}_j^k \; \forall\, j, k$. The set of such combinations, X^{**}, is then defined as

$$X^{**} = \{\mathbf{x}^{**} \in X^* : \nexists\, \mathbf{x} \in X^* \text{such that } \mathbf{x} \text{ dominates } \mathbf{x}^{**}\}$$

We define **ommitted bidders** as those for whom there is at least one feasible outcome in X^{**} which does not include a bid for the bidder. Thus, the set of ommitted bidders, θ, is:

$$\theta = \left\{ j \in \{1, 2, \ldots, N\} : \; \exists\, \mathbf{x} \in X^{**}, \; \sum_{k=1}^{K_j} x_j^k = 0 \right\}$$

Let B denote the set of all the bids received in the second auction stage.

Let β_j^h denote the headline bid submitted by bidder j in the most recent round with lot prices being $\mathbf{p} = (p_{A1}, p_{A2}, p_{A3}, p_B, p_C, p_D, p_E, p_F)$, and let $\Lambda^j = \{\lambda \in \{A1, A2, A3, B, C, D, E, F\} : q_\lambda \text{ in } \mathbf{q}_j^h > 0\}$, i.e. the set of lot categories for which the bidder has specified positive demand in its most recent headline bid.

In order to identify the lot categories for which prices need to increase, for each omitted bidder $n \in \theta$ we run through the following steps:

1. For each lot category $\lambda \in \Lambda^n$ we do the following:
   (a) construct a hypothetical set of bids $\tilde{B}$, which contains all the bids received in the second auction stage except for β_n^h (bidder n's headline bid), which we replace with a hypothetical headline bid $\beta_j^{\tilde{h}} = (P_j^{\tilde{h}}; \mathbf{q}_j^{\tilde{h}})$ where q_λ is the same as in the original headline bid, and the quantity of all other lots is zero, and thus the bid amount is $P_j^{\tilde{h}} = q_\lambda \cdot p_\lambda$. Thus $\tilde{B} = \left\{ \beta_j^{\tilde{h}} \cup B \setminus \beta_n^h \right\}$.
   (b) Identify the value-maximising feasible combinations when considering $\tilde{B}$. If bidder n would still be omitted, then we identify λ as a lot category that requires a price increment.
2. If after running step 1 for bidder n for all lot categories we have not identified any lot category that requires a price increment, then all the lot categories in Λ^n will require a price increment.

We can terminate this process without having to run through the steps for all omitted bidders if we find that all lot categories require a price increment.