Detailed description of the closing condition and the process for establishing the need for price increases

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1 The closing condition

The third stage of the auction closes when there is at least one Potentially Winning Combination of Bids that includes a bid from every bidder.

Let \( B \) denote the number of B lots available in the third auction stage. The number of lots in each exemption lot category included in the third stage of the auction is equal to the number of winners of B lots minus one, or zero if there are no winners of B lots.

Let \( \beta_j^k = (P_j^k, b_j^k, c_1^j, c_2^j, c_3^j) \) be a bid \( k \) from bidder \( j = 1, \ldots, N \), where \( N \) is the number of bidders taking part in the third auction stage, \( P_j^k \) is the amount of the bid, \( b_j^k \) is the number of B lots included in the package of the bid, and \( c_j^c \) for \( c = c_1, c_2, c_3 \) indicates whether the package includes an exemption for the respective Coverage Area Group (depending on the number of Coverage Area Groups that are included in the third auction stage). \( K_j \) denotes the number of bids submitted by bidder \( j \).

A combination of bids is defined by \( x = (x_1^1, x_2^1, \ldots, x_1^{K_1}, x_2^1, \ldots, x_N^{K_N}) \), where \( x_j^k \) is a binary variable that is equal to 1 if bidder \( j \)'s \( k \)-th bid is included in the combination and zero otherwise.

In order to check whether the closing condition is met, we define the following:

A **Supply Scenario** \( S_i \) with \( i = 0, \ldots, N - 1 \) is a scenario in which the supply of exemptions for each Coverage Area Group that is included in the third auction stage is fixed at \( i \). With \( N \) bidders taking part in the third auction stage, \( N - 1 \) is the maximum number of exemptions that might be available.

A **Feasible Combination of Bids** in a Supply Scenario \( S_i \) is a combination of bids that includes at most one bid from each bidder and can be accommodated with the given supply in that scenario. Thus, a Feasible Combination of Bids in Supply Scenario \( S_i \) is defined as a combination of bids that satisfies the following conditions:

\[
\sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k b_j^k \leq B \quad (1.1)
\]
\[
\sum_{j=1}^{N} \sum_{k=1}^{K_j} x_{jk}^{k} \leq i \quad \forall \ c = c1, c2, c3
\] (1.2)

\[
\sum_{k=1}^{K_j} x_{jk}^{k} \leq 1 \quad \forall \ j = 1, ..., N
\] (1.3)

The Value of a Feasible Combination of Bids is denoted as

\[
V(x) = \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_{jk}^{k} P_{jk} + \left( B - \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_{jk}^{k} b_{jk} \right) r
\]

where \( r \) is the reserve price of B lots.

The set \( X_i^* \) of Highest-Value Combinations of Bids \( x_i^* \) in a Supply Scenario \( S_i \) is the set of Feasible Combinations of Bids that achieve the highest value, i.e.

\[
X_i^* = \left\{ x_i^* : \arg \max_x V(x) \mid S_i \right\}
\]

subject to conditions 1.1 to 1.3.

A Feasible Combination of Bids \( x_i \) in a Supply Scenario \( S_i \) is Exemption-Compatibility if:

\[
\max \left[ \left( \sum_{j=1}^{N} z_{ij} x_{ij}^k \right) - 1, 0 \right] \geq i
\]

where \( z_{ij}^k = 1 \) if the bid from bidder \( j \) includes one or more B lots, and zero otherwise.

Let \( \hat{X}_i \) denote the set of all Combinations of Bids in Supply Scenario \( S_i \) that are Exemption-Compatible.

Let \( \hat{X}_i^* \) denote the set of the Highest-Value Combination of Bids in Supply Scenario \( S_i \) that are Exemption-Compatible, so \( \hat{X}_i^* = X_i^* \cap \hat{X}_i \forall i = 0, ..., N - 1 \).

Let \( \hat{X}^* \) denote the set of all the Exemption-Compatible, Highest-Value Combination of Bids across all scenarios, so \( \hat{X}^* = \bigcup_{i=0}^{N-1} \hat{X}_i^* \).

The Potentially Winning Combinations of Bids \( X^* \) is the set of Exemption-Compatible Highest-Value Combinations of Bids across all Supply Scenarios with the highest value, i.e.

\[
X^* = \left\{ \hat{x}_i^* : \arg \max_x V(\hat{x}_i^*) \mid \hat{x} \in \hat{X}^* \right\}
\]

Using these definitions, the third stage of the auction closes when the following conditions hold:

\[
\exists x \in X^* : \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_{jk}^{k} = N
\]
2 Identifying lot categories for which prices need to increase

If the third auction stage does not close, each of the Potential Winning Combinations of Bids omits all bids from at least one bidder. We call such a bidder an Omitted Bidder.

In order to identify Omitted Bidders, we divide the bidders into two distinct sets:

- The first set contains all bidders who are included in each and every Potentially Winning Combination.
- All other bidders fall into the second set (the set of Omitted Bidders).

For notational convenience, let us re-label bidders so that all bidders \( j = 1, \ldots, J - 1 \) are in the first set, and all bidders \( j = J, \ldots, N \) are in the second set, i.e. are Omitted Bidders. The order of bidders within each set is random.

Let \( \beta^h_j \) denote the headline bid submitted by bidder \( j \) in the most recent round with lot prices being \( p_b \) and \( p_c \) with \( c = c_1, c_2, c_3 \).

In order to identify the lot categories for which prices need to increase, we examine for each omitted bidder \( n = J, \ldots, N \) the potential reason why that bidder’s headline bid could not be accommodated. Specifically, we run through the following steps (terminating the process early if we arrive at a point at which we have identified all lot categories as needing a price increase):

1. Set \( n = J \).
2. Identify the Potentially Winning Combinations of Bids having replaced \( \beta^h_n \) with a hypothetical bid \( \beta' = (P^h_n - \sum c^h_n p_c, b^h_n, 0, 0, 0) \). If bidder \( n \) would still be omitted, the price of B lots needs to increase.
3. Considering each of the available exemption lot categories \( c_1, c_2, c_3 \) for which \( \beta^h_n = 1 \), in turn:
   (a) identify the Potentially Winning Combinations of Bids having replaced \( \beta^h_n \) with a hypothetical bid \( \beta' \) in which the respective \( \beta^h_n \) is retained, the demand for all other lot categories is set to zero and the bid amount is reduced accordingly;
   (b) if bidder \( n \) would still be omitted, the price of exemption lots in category \( c \) needs to increase.
4. If \( \beta^h_n \) includes bids for more than one exemption lot category and neither the first nor the second step have identified the need for a price increase, identify the Potentially Winning Combinations of Bids having replaced \( \beta^h_n \) with a hypothetical bid \( \beta' = (P^h_n - p_b, 0, c_1^h_n, c_2^h_n, c_3^h_n) \). If bidder \( n \) would still be omitted, the price of exemption lots in all categories included in the bidder’s headline bid needs to increase.
5. If after steps 2 to 4, none of the lot categories have been identified as requiring a price increase, the prices of B lots and exemptions in all Coverage Area Groups for which \( \beta^h_n = 1 \) need to increase.
6. If \( n < N \), set \( n = n + 1 \) and repeat steps 2 to 5 and perform the tests if the corresponding lot category has not already been identified as requiring a price increase. Otherwise, terminate.